

# The ABC's of affine Grassmannians and Hall-Littlewood polynomials

Avinash J. Dalal<sup>†</sup> and Jennifer Morse<sup>‡</sup>

Department of Mathematics, Drexel University, Philadelphia, PA 19104, U.S.A.

---

**Abstract.** We give a new description of the Pieri rule for  $k$ -Schur functions using the Bruhat order on the affine type- $A$  Weyl group. In doing so, we prove a new combinatorial formula for representatives of the Schubert classes for the cohomology of affine Grassmannians. We show how new combinatorics involved in our formulas gives the Kostka-Foulkes polynomials and discuss how this can be applied to study the transition matrices between Hall-Littlewood and  $k$ -Schur functions.

Nous présentons une nouvelle description, issue de l'ordre de Bruhat du groupe de Weyl affine de type  $A$ , de la règle de Pieri pour les fonctions  $k$ -Schur. Ce faisant, nous obtenons une nouvelle formule combinatoire pour les représentants des classes de Schubert de la cohomologie des Grassmanniennes affines. Nous décrivons aussi comment notre approche permet d'obtenir les polynômes de Kostka-Foulkes et comment elle peut être appliquée à l'étude des matrices de transition entre les polynômes de Hall-Littlewood et les fonctions  $k$ -Schur.

**Keywords:**  $k$ -Schur functions, Pieri rule, Bruhat order, Macdonald polynomials, Hall-Littlewood polynomials,  $k$ -tableaux

---

## 1 Introduction

The dual  $k$ -Schur functions arose in [LM08] where it was shown that their coproduct encodes structure constants of the Verlinde fusion algebra for the  $\widehat{\mathfrak{sl}}_n$  Wess–Zumino–Witten models [TUY89, Ver88] or equivalently, the 3-point Gromov–Witten invariants of genus zero (e.g. [Wit95]). It was further proven in [Lam08] that these functions represent cohomology classes of the affine Grassmannian. Dual  $k$ -Schur functions are defined as the weight generating functions of  $k$ -tableaux; a combinatorial object encoding successions of saturated chains in weak order on the affine Weyl group  $\tilde{A}^{k+1}$  (see § 2). Here, we find that there is a natural realization of weak saturated chains of length  $\ell$  as length  $k - \ell$  saturated chains in the strong (Bruhat) order on  $\tilde{A}^{k+1}$ . This enables us to prove that the dual  $k$ -Schur functions  $\mathfrak{S}_\lambda^{(k)}$  can be described in terms of a new combinatorial object called affine Bruhat counter-tableau, or *ABC*, that encodes successions of strong chains (see Definition 16). We prove, for a fixed positive integer  $k$  and  $k + 1$ -core  $\lambda$  that

$$\mathfrak{S}_\lambda^{(k)} = \sum_A x^A, \quad (1)$$

---

<sup>†</sup>Partially supported by the NSF grants DMS–0652641, DMS–0638625, DMS–1001898.

<sup>‡</sup>Partially supported by the NSF grants DMS–0652641, DMS–0638652.

over the  $ABC$ 's of inner shape  $\lambda$ .

Our new interpretation for  $\{\mathfrak{S}_\lambda^{(k)}\}_{\lambda_1 \leq k}$  puts these cohomology classes in the same combinatorial vein as the homology classes. That is, representatives for the homology classes of the affine Grassmannian are given by the  $k$ -Schur functions  $s_\lambda^{(k)}$  of [LM07] which were defined in terms of chains in the strong order on  $\tilde{A}^{k+1}$  by [LLMS10]. In fact, our work relies on a reformulation of the Pieri rule for  $s_\lambda^{(k)}$ ; we show that this rule for computing the  $k$ -Schur expansion of the product  $h_r s_\lambda^{(k)}$  can be described by certain saturated strong chains.

A strong motivation for introducing these new combinatorial ideas is to shed light on open problems in the theory of Macdonald polynomials. The origin of  $k$ -Schur functions was in the idea that there exists a more refined basis than Schur functions upon which Macdonald polynomials  $H_\mu[X; q, t]$  expand positively. In particular, a family of polynomials  $A_\lambda^{(k)}[X; t]$  was introduced by [LLM03] and it was conjectured that, for  $\mu_1 \leq k$ ,

$$H_\mu[X; q, t] = \sum_{\lambda} K_{\lambda\mu}^{(k)}(q, t) A_\lambda^{(k)}[X; t] \quad (2)$$

and  $K_{\lambda\mu}^{(k)}(q, t) \in \mathbb{N}[q, t]$ . When  $k \geq |\lambda|$ ,  $A_\lambda^{(k)}[X; t]$  reduces to the Schur function  $s_\lambda$ . In this case, the conjecture reduces to the conjecture of [Mac88] that  $K_{\lambda\mu}^{(\infty)}(q, t)$  is a positive sum of monomials in  $q$  and  $t$ . The polynomiality of  $K_{\lambda\mu}^{(\infty)}(q, t)$  was proven independently by [KN98, Sah96, Kno97, GT96, LV97] and the positivity was finally settled by [Hai01]. However, for generic  $k$ , this remains an open problem even in the Hall-Littlewood case  $H_\mu[X; 0, t]$ .

It is conjectured that  $A_\mu^{(k)}[X; 1]$  are the  $k$ -Schur functions  $s_\mu^{(k)}$  prompting our study of  $ABC$  combinatorics in the context of Hall-Littlewood polynomials. To this end, we introduce a simple spin statistic in § 5 on  $ABC$ 's and conjecture that the  $K_{\lambda\mu}^{(k)}(0, t)$  can be described as the spin generating function of  $ABC$ 's. To provide evidence for our conjecture, we prove that the [LS78] formula for Kostka-Foulkes polynomials in terms of charge is equivalent to

$$K_{\lambda\mu}(t) = \sum_A t^{\text{spin}(A)},$$

over all  $ABC$ 's of weight  $\mu$  and inner shape  $\lambda$ .

## 2 $k$ -Schur Functions

Dual  $k$ -Schur functions are defined as the weight generating functions of  $k$ -tableaux. To understand the definition of these tableaux, let us set some notation. We identify each partition  $\lambda = (\lambda_1, \dots, \lambda_n)$  with its Ferrers shape (having  $\lambda_i$  lattice squares in the  $i^{\text{th}}$  row, from the bottom to top). A column-strict tableau is a filling of a shape with positive integers that weakly decrease along rows and strictly increase up columns. A  $p$ -core is a partition that does not contain any cell with hook-length  $p$ . We use  $\mathcal{C}^p$  to denote the set of all  $p$ -cores. The content of cell  $(i, j)$  is  $j - i$ , and its  $p$ -residue is  $j - i \bmod p$ . The 5 residues of the 5-core  $(6, 4, 3, 1, 1, 1)$  are

0
1
2
3 4 0
4 0 1 2
0 1 2 3 4 0

**Definition 1** Let  $\nu \in \mathcal{C}^{k+}$  and let  $\alpha = (\alpha_1, \dots, \alpha_r)$  be a composition of  $\deg(\nu)$ . A  $k$ -tableau of shape  $\nu$  and  $k$ -weight  $\alpha$  is a column-strict filling of  $\nu$  with integers  $1, 2, \dots, r$  such that the collection of cells filled with letter  $i$  are labeled by exactly  $\alpha_i$  distinct  $k+1$ -residues for  $1 \leq i \leq r$ .

3 <sub>5</sub>	3 <sub>6</sub>						
2 <sub>6</sub>	2 <sub>0</sub>						
1 <sub>0</sub>	1 <sub>1</sub>	1 <sub>2</sub>	2 <sub>3</sub>	3 <sub>4</sub>	3 <sub>5</sub>	3 <sub>6</sub>	4 <sub>0</sub>

3 <sub>5</sub>	4 <sub>6</sub>						
2 <sub>6</sub>	3 <sub>0</sub>						
1 <sub>0</sub>	1 <sub>1</sub>	1 <sub>2</sub>	2 <sub>3</sub>	2 <sub>4</sub>	3 <sub>5</sub>	3 <sub>6</sub>	3 <sub>0</sub>

$$\mathfrak{S}_\lambda^{(k)} = \sum_{\substack{T: k\text{-tableau} \\ \text{shape}(T) = \lambda}} x^{k\text{-weight}(T)}. \quad (3)$$
$$h_\mu = \sum_\lambda K_{\lambda\mu}^{(k)} s_\lambda^{(k)} \quad (5)$$

should reveal the  $k$ -tableaux. In fact, starting with  $s_{\emptyset}^{(k)} = 1$ ,  $k$ -tableaux are precisely the objects that encode the combinatorial Pieri rule for  $k$ -Schur functions given in [LM07]. That is, for  $\lambda \in \mathcal{C}^{k+1}$  and  $0 < \ell \leq k$ ,

$$h_{\ell} s_{\lambda}^{(k)} = \sum_{\nu \in H_{\lambda, \ell}^{(k)}} s_{\nu}^{(k)},$$

over a specified set of cores  $H_{\lambda, \ell}^{(k)}$  described by certain saturated chains in the weak order on the affine Weyl group  $\tilde{A}^{k+1}$ . Recall that weak order on  $\tilde{A}^{k+1}$  can be realized on cores by the covering relation

$$\beta \triangleleft \nu \iff \nu = \beta + \text{all addable corners of one fixed residue.}$$

**Definition 2** For  $0 < \ell \leq k$  and  $k+1$ -cores  $\lambda$  and  $\nu$ , the skew shape  $\nu/\lambda$  is a weak  $\ell$ -strip if there is a weak saturated chain of cores

$$\lambda = \gamma^0 \triangleleft \gamma^1 \triangleleft \dots \triangleleft \gamma^{\ell} = \nu,$$

where  $\nu/\lambda$  is a horizontal strip and the rightmost cell of  $\gamma^i/\gamma^{i-1}$  is to the left of that in  $\gamma^{i+1}/\gamma^i$  for all  $i = 1, \dots, \ell - 1$ .

It should be observed that if there exists in the previous definition such a weak saturated chain then it is unique.

**Example 3** The skew shape  $(4, 1, 1, 1)/(3, 1, 1)$  of 4-cores is a weak 2-strip as there is the saturated chain

The terms appearing in the summand of the  $k$ -Pieri rule are simply the set of weak  $\ell$ -strips. That is,

$$H_{\lambda, \ell}^{(k)} = \{\nu : \nu/\lambda \text{ is a weak } \ell\text{-strip}\}.$$

## 2.2 Strong $k$ -Pieri Rule

Recall that the strong (Bruhat) order on affine type  $A$  is realized on cores by the covering relation:

$$\rho \triangleleft_B \gamma \iff \rho \subseteq \gamma \text{ and } \deg(\gamma) = \deg(\rho) + 1.$$

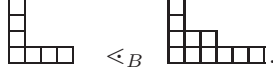
An important fact about strong covers is useful in our study. An  $n$ -ribbon  $R$  is a skew diagram  $\lambda/\mu$  consisting of  $n$  rookwise connected cells such that there is no  $2 \times 2$  shape contained in  $R$ . We refer to the southeasternmost cell of a ribbon as its *head*.

**Lemma 4** [LLMS10] Let  $\rho \triangleleft_B \gamma$  be cores. Then

1. Each connected component of  $\rho/\gamma$  is a ribbon.
2. The components are translates of each other and their heads have the same residue.

**Definition 5**  $(\gamma, c)$  is a marked strong cover of a  $k+1$ -core  $\rho$  if  $\rho \triangleleft_B \gamma$  and  $c$  is the content of the head of a ribbon in  $\gamma/\rho$ .

**Example 6** Consider the 4-cores  $\rho = (4, 1, 1, 1), \gamma = (6, 3, 1, 1)$ .



$(\gamma, 5)$  is a marked strong cover of  $\rho$ , as is  $(\gamma, 1)$ .

**Definition 7** For  $0 < \ell \leq k$  and  $k+1$ -cores  $\lambda$  and  $\gamma$ , a strong  $\ell$ -strip from  $\lambda$  to  $\gamma$  is a saturated chain in the strong order

$$\lambda = \gamma^0 \leq_B \gamma^1 \leq_B \dots \leq_B \gamma^\ell = \gamma$$

together with a content vector  $c = (c_1, c_2, \dots, c_\ell)$  such that

1.  $(\gamma^i, c_i)$  is a marked strong cover of  $\gamma^{i-1}$ ,
2.  $c_{i-1} < c_i$  for  $2 \leq i \leq \ell$ .

**Example 8** The following sequence of marked strong covers is a strong 2-strip from  $(1)$  to  $(3, 1)$



since the content vector  $c = (1, 2)$  is increasing.

The strong  $\ell$ -strips help to encode the combinatorial strong Pieri rule for the dual  $k$ -Schur functions. That is, for  $\lambda \in \mathcal{C}^{k+1}$  and  $0 < \ell \leq k$ ,

$$h_\ell \mathfrak{S}_\lambda^{(k)} = \sum_{\gamma} d_\gamma \mathfrak{S}_\gamma^{(k)},$$

where  $d_\gamma$  is the number of strong  $\ell$ -strips from  $\lambda$  to  $\gamma$ .

### 3 Bottom Strong Strips

We have discovered that there is a natural association between weak strips and certain strong strips. In this association, a saturated chain in weak order of length  $\ell$  is identified with a saturated chain in strong order of length  $k - \ell$ .

Our notion comes from a close examination of the core  $(k + \lambda_1, \lambda)$  constructed from any  $\lambda \in \mathcal{C}^{k+1}$ .

**Definition 9** For  $0 < \ell \leq k$  and  $k+1$ -cores  $\lambda$  and  $\nu$ , the skew shape  $(k + \lambda_1, \lambda)/\nu$  is a bottom strong  $\ell$ -strip if there is a saturated chain of cores

$$\nu = \nu^0 \leq_B \nu^1 \leq_B \dots \leq_B \nu^{k-\ell} = (k + \lambda_1, \lambda),$$

where

1.  $(k + \lambda_1, \lambda)/\nu$  is a horizontal strip.
2. The bottom rightmost cell of  $\nu^i$  (which is always in the first row) is also a cell in  $\nu^i/\nu^{i-1}$ , for  $1 \leq i \leq k - \ell$ .

It turns out that in the previous definition if such a chain exists then it is unique.

**Remark 10** Every bottom strong  $\ell$ -strip  $(k + \lambda_1, \lambda)/\nu$  is a strong  $\ell$ -strip if we take the content of  $\nu^i$  to be the content of its bottom rightmost cell in the corresponding saturated chain of cores

$$\nu = \nu^0 \leq_B \nu^1 \leq_B \cdots \leq_B \nu^{k-\ell} = (k + \lambda_1, \lambda).$$

**Example 11** The skew shape  $(8, 3)/(4, 2)$  of 6-cores is a bottom strong 3-strip as there is the saturated chain

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \leq_B \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square \\ \hline \end{array} \leq_B \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \hline \end{array}.$$

**Example 12** The skew shape  $(6, 3, 1, 1)/(4, 1, 1, 1)$  of 4-cores is a bottom strong 2-strip as there is the saturated chain

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \leq_B \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}.$$

This is an example of a bottom strong 2-strip which is also a weak 2-strip (compare to Example 3). In fact, there is a close connection between bottom  $\ell$ -strips and weak  $\ell$ -strips.

**Theorem 13** For  $0 < \ell \leq k$  and  $k + 1$ -cores  $\lambda$  and  $\nu$ ,

$$(k + \lambda_1, \lambda)/\nu \text{ is a bottom strong } \ell\text{-strip} \iff \nu/\lambda \text{ is a weak } \ell\text{-strip}.$$

One immediate application of this result is that the  $k$ -Pieri rule can instead be given in terms of bottom strong  $\ell$ -strips.

**Corollary 14** For any  $0 < \ell \leq k$  and  $\lambda \in \mathcal{C}^{k+1}$ ,

$$h_\ell s_\lambda^{(k)} = \sum_{\substack{\nu \in \mathcal{C}^{k+1} \\ (k + \lambda_1, \lambda)/\nu: \text{bottom strong } \ell\text{-strip}}} s_\nu^{(k)}.$$

**Example 15** When  $k = 3$ , the set of  $\nu$  such that  $(6, 3, 1, 1)/\nu$  is a bottom strong 2-strip is

$$\{(3, 3, 1, 1), (5, 2, 1), (4, 1, 1, 1)\},$$

and thus

$$h_2 s_{(3,1,1)}^{(3)} = s_{(3,3,1,1)}^{(3)} + s_{(5,2,1)}^{(3)} + s_{(4,1,1,1)}^{(3)}.$$

## 4 Affine Bruhat Counter-Tableaux

The iteration of Corollary 14 gives rise to a new combinatorial structure whose enumeration describes the coefficients in

$$h_\lambda = \sum_{\nu} K_{\nu\lambda}^{(k)} s_\nu^{(k)}. \quad (6)$$

To describe this structure, first let us set more notations. Recall that a counter-tableau  $A$  is a filling of a skew shape with numbers that strictly decrease up columns and weakly decrease along the rows. Given a counter-tableau  $A$ , let  $A^{(x)}$  denote the subtableau made up of the rows of  $A$  weakly higher than row  $x$  (where row 1 is this time the *highest* row). Let  $A_{>i}$  denote the restriction of  $A$  to letters strictly larger than  $i$  where empty cells in a skew are considered to contain  $\infty$ .

$(k + \lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)}$  is a bottom strong  $\alpha_i$ -strip for all  $1 \leq i \leq \ell(\alpha)$ .

In the construction of a counter-tableau, we start with the empty shape  $\lambda^{(0)}$ , successively adding bottom strong strips and tiling  $(k + \lambda_1^{(i-1)}, \lambda^{(i-1)})/\lambda^{(i)}$  with  $i$ -ribbons at each step.

3	3	2	1	1					
			3	2	2	2	2		
				3	3	3	3	3	

*strong 3-strip* :  $\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array} \triangleleft_B \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline\end{array} \triangleleft_B \begin{array}{|c|c|c|c|c|}\hline & & & 1 & 1 \\ \hline\end{array}$

strong 3-strip : 

strong 1-strip :

For a composition  $\alpha$  whose entries are not larger than  $k$ , there is a known bijection between the set of all standard  $k$ -tableaux of  $k$ -weight  $\alpha$  and the set of all  $ABC$ 's of  $k$ -weight  $\alpha$ . Letting  $K_{\nu\lambda}^{(k)}$  denote the number of affine Bruhat counter-tableaux of inner shape  $\nu$  and  $k$ -weight  $\lambda$ , it is not hard to see that it is this set of  $ABC$ 's being enumerated by the coefficients in (6) and by duality we can prove that the dual  $k$ -Schur functions are the  $ABC$  generating functions.

$$\mathfrak{S}_\lambda^{(k)} = \sum_A x^A$$

## 5 Hall-Littlewood Expansions

$$K_{\lambda\mu}^{(k)}(1,1) = K_{\lambda 1^n}^{(k)}$$

are the coefficients in (6) when  $\mu = (1^n)$ . We have seen that these count the number of  $ABC$ 's of shape  $\lambda$  with  $k$ -weight  $(1^n)$  (called *standard ABC*'s). Assuming the positivity of  $K_{\lambda\mu}^{(k)}(q, t)$ , this leads us to

believe there exists a pair of statistics (non-negative integers)  $a(A)$  and  $b(A)$  associated to each  $ABC$  so that

$$K_{\lambda\mu}^{(k)}(q, t) = \sum_{\substack{A=\text{standard } ABC \\ \text{inner shape}(A)=\lambda}} t^{a_\mu(A)} q^{b_\mu(A)}.$$

As a special case, when  $k = n$ , these statistics would give a combinatorial formula for the  $q, t$ -Kostka polynomials arising in the Schur expansion of the Macdonald polynomials – a long-standing open problem in the field. Setting  $q = 0$ , the Macdonald polynomials reduce to Hall-Littlewood polynomials. In fact, when  $k = n$ , [LS78] give a beautiful combinatorial formula for the Kostka-Foulkes polynomials  $K_{\lambda\mu}^{(n)}(0, t)$  by defining a statistic on column-strict tableaux. It is interesting to note that  $k$ -tableaux reduce to column-strict tableaux when  $k = n$ , whereas the  $ABC$ 's are truly of a different spirit even when  $k = n$ . We thus start by introducing a statistic on  $ABC$ 's when  $k = n$ .

Here, we will restrict our attention to *standard*  $ABC$ 's – those of weight  $(1^n)$ . It turns out that a standard  $ABC$  has only 1 or 2-ribbons. In fact, when  $k = n$ , there is a simple statistic that gives the Kostka-Foulkes polynomials.

**Definition 19** For each  $ABC$   $A$  of  $n$ -weight  $1^n$ , let

$$\text{spin}(A) = \sum_i i \chi(\text{row } i \text{ has a 2-ribbon and it is not east of a 2-ribbon in row } i+1),$$

where row 1 is the bottommost row of  $A$ , and where  $\chi$  is the indicator function evaluating to zero when the statement is false and to 1 otherwise. Recall that a ribbon is east of another ribbon if the head of the former is to the east of the latter's.

**Example 20** Consider the  $ABC$  of 5-weight  $1^5$

$$A = \begin{array}{|c|c|c|c|c|c|} \hline 2 & 1 & 1 & 1 & 1 & \\ \hline 4 & 3 & 2 & 2 & 2 & 2 \\ \hline & 5 & 3 & 3 & \bar{3} & \bar{3} & 3 \\ \hline & & 4 & 4 & 4 & \bar{4} & \bar{4} \\ \hline & & & 5 & \bar{5} & \bar{5} & 5 & 5 \\ \hline \end{array}.$$

2-ribbons are colored (or for those without color, 2-ribbons are depicted with a bar). Of these 3 2-ribbons, only the one in row 1 and row 3 contribute to the spin since the 2-ribbon in row 2 is east of that in row 3. The spin is thus  $1+3=4$ .

**Theorem 21** For  $\lambda \vdash n$ ,

$$K_{\lambda, 1^n}(0, t) = K_{\lambda 1^n}^{(n)}(0, t) = \sum_{\substack{A: ABC \text{ of } n\text{-weight } 1^n \\ \text{inner shape}(A)=\lambda}} t^{\text{spin}(A)}. \quad (7)$$

**Example 22** The set of all  $ABC$ 's of 3-weight  $1^3$  are

$$\left\{ \begin{array}{|c|c|c|} \hline & 1 & 1 \\ \hline & 2 & \bar{2} & \bar{2} \\ \hline & \bar{3} & \bar{3} & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 1 & 1 \\ \hline & 2 & \bar{2} & \bar{2} \\ \hline & & 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline & 3 & 2 & 2 \\ \hline & & \bar{3} & \bar{3} & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 2 \\ \hline & & & 3 & 3 \\ \hline \end{array} \right\}$$



The spins of these ABC's, respectively, are  $\{3, 2, 1, 0\}$ . We thus have that

$$H_{1^3}[X, t] = t^3 s_{(1,1,1)} + (t^2 + t) s_{(2,1)} + s_{(3)}.$$

The beauty of this formulation is that it gives a hint into finding a statistic to describe the more general Hall-Littlewood coefficients  $K_{\lambda, 1^n}^{(k)}(t)$ . To do this, we need only the extra concept of an *offset* in an ABC. In an ABC, for  $i > 1$ , an  $i$ -ribbon  $R$  is an *offset* if there is a lower  $i$ -ribbon filled with the same letter as  $R$  whose head has the same residue as the head of  $R$ . The number of offsets in the ABC is denoted  $\text{off}(ABC)$ .

**Example 23** Consider the ABC of 3-weight  $1^5$ .

$$A = \begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 1 & & \\ \hline & 2 & \bar{2} & \bar{2} & \\ \hline & 4 & 3 & 3 & \\ \hline & \bar{5} & \bar{5} & 4 & 4 \\ \hline & & & 5 & \bar{5} & \bar{5} \\ \hline \end{array}$$

Here we see that  $A$  has only one offset  $\begin{bmatrix} \bar{5} & \bar{5} \end{bmatrix}$  in the second row from the bottom. Observe that it is an offset since there is another 2-ribbon of the same residue in the first row with the same letter. This tells us that  $\text{off}(A) = 1$ .

**Definition 24** Let  $A$  be an ABC of  $k$ -weight  $1^n$  and define

$$\text{spin}^k(A) = \text{off}(A) + \sum_i i \chi(\text{row } i \text{ has a 2-ribbon and it is not east of a 2-ribbon in row } i + 1).$$

When  $k = n$ , an ABC never has any offsets and thus the above definition reduces to Definition 19.

**Conjecture 25** For any  $k + 1$ -core  $\lambda$  where  $\deg(\lambda) = n$ ,

$$K_{\lambda 1^n}^{(k)}(0, t) = \sum_{\substack{A: \text{ABC of weight } 1^n \\ \text{inner shape}(A) = \lambda}} t^{\text{spin}^k(A)}. \quad (8)$$

**Example 26** The set of all ABC's of 2-weight  $1^4$  are

$$\left\{ \begin{array}{|c|c|c|c|} \hline & 1 & & \\ \hline & \bar{2} & \bar{2} & \\ \hline & & 3 & \\ \hline & & \bar{4} & \bar{4} \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 4 & 1 & & \\ \hline & \bar{2} & \bar{2} & \\ \hline & 4 & 3 & \\ \hline & & & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & & & \\ \hline & 3 & 2 & & \\ \hline & \bar{4} & \bar{4} & 3 & \\ \hline & & & \bar{4} & \bar{4} \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|c|} \hline 2 & 1 & & & \\ \hline 4 & 3 & 2 & & \\ \hline & & 4 & 3 & \\ \hline & & & & 4 \\ \hline \end{array} \right\}$$

Hence, our conjecture checks out against the expansion of  $H_{1^4}[X, t]$  in terms of 2-Schur functions:

$$H_{1^4}[X, t] = s_{(2,2,1,1)}^{(2)} + t^2 s_{(3,1,1)}^{(2)} + t^3 s_{(3,1,1)}^{(2)} + t^4 s_{(4,2)}^{(2)}.$$

## 6 Related and Future work

Central to the proof of the [LS78] formula for Hall-Littlewood polynomials is the *plactic monoid*, [LS81]. The plactic monoid is the quotient of the free monoid  $A^*$  on the totally ordered set  $A$  by *Knuth relations*, defined on words so  $w \sim w'$  if and only if they are sent to the same tableau under RSK-insertion (see [Knu70]). This establishes a bijection  $T \mapsto [w] \in (A^* / \sim)$  and the monoid can be viewed as the set of tableaux on letters in  $A$  with multiplication defined by insertion. The ring of symmetric functions  $\Lambda$  can be identified with a subring of the monoid algebra  $\mathbb{Z}[q, t][A^* / \sim]$  by sending the Schur function  $s_\lambda$  to the sum of all tableaux with shape  $\lambda$ . Although the plactic monoid is noncommutative, the combinatorial nature of computation is more evident.

A refinement of the plactic monoid to a structure on  $k$ -tableaux that can be applied to combinatorial problems involving  $k$ -Schur functions is partially given in [LLMS] by a bijection compatible with the RSK-bijection. A deeper understanding of this highly intricate bijection is underway. Towards this effort, Lapointe and Pinto have discovered a statistic on  $k$ -tableaux that is compatible with the bijection [LP]. We conjecture that their statistic matches the spin of our  $ABC$ 's and are working to put  $ABC$ 's in a context that simplifies the bijection.

## References

- [Ber09] François Bergeron. *Algebraic combinatorics and coinvariant spaces*. CMS Treatises in Mathematics. Canadian Mathematical Society, Ottawa, ON, 2009.
- [GT96] A. M. Garsia and G. Tesler. Plethystic formulas for Macdonald  $q, t$ -Kostka coefficients. *Adv. Math.*, 123:144–222, 1996.
- [Hai01] M. Haiman. Hilbert schemes, polygraphs, and the macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006, 2001.
- [KN98] A. N. Kirillov and M. Noumi. Affine hecke algebras and raising operators for macdonald polynomials. *Duke Math. J.*, 93(1):1–39, 1998.
- [Kno97] F. Knop. Integrality of two variable Kostka functions. *J. Reine Angew. Math.*, 482:177–189, 1997.
- [Knu70] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.
- [Lam06] T. Lam. Affine stanley symmetric functions. *Amer. J. Math.*, 128(6):1553–1586, 2006.
- [Lam08] T. Lam. Schubert polynomials for the affine grassmannian. *J. of Amer. Math. Soc.*, 21(1):259–281, 2008.
- [LLM03] L. Lapointe, A. Lascoux, and J. Morse. Tableau atoms and a new macdonald positivity conjecture. *Duke Math. J.*, 116(1):103–146, 2003.
- [LLMS] T. Lam, L. Lapointe, J. Morse, and M. Shimozono. The poset of  $k$ -shape and branching of  $k$ -schur functions. *to be published in Mem. Amer. Math. Soc.*

- [LLMS10] T. Lam, L. Lapointe, J. Morse, and M. Shimozono. Affine insertion and pieri rules for the affine grassmannian. *Mem. Amer. Math Soc.*, 208(977), 2010.
- [LM05] L. Lapointe and J. Morse. Tableaux on  $k + 1$ -cores, reduced words for affine permutations, and  $k$ -schur function expansions. *J. Combin. Theory Ser.*, 112:44–81, 2005.
- [LM07] L. Lapointe and J. Morse. A  $k$ -tableaux characterization for  $k$ -schur functions. *Adv. Math.*, pages 183–204, 2007.
- [LM08] L. Lapointe and J. Morse. Quantum cohomology and the  $k$ -schur basis. *Trans. Amer. Math. Soc.*, 360:2021–2040, 2008.
- [LP] L. Lapointe and M. E. Pinto. Private communication.
- [LS78] A. Lascoux and M.-P. Schützenberger. Sur une conjecture de h.o. foulkes. *C.R. Acad. Sc. Paris*, 294:323–324, 1978.
- [LS81] A. Lascoux and M.-P. Schützenberger. Le monoïde plaxique. *Quaderni della Ricerca scientifica*, 109:129–156, 1981.
- [LV97] L. Lapointe and L. Vinet. A rodrigues formula for the macdonald polynomials. *Adv. Math.*, 130:261–279, 1997.
- [Mac88] I. G. Macdonald. A new class of symmetric functions. *Séminaire Lotharingien de Combinatoire*, **B20a** 41pp, 1988.
- [Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Clarendon Press, Oxford, 2nd edition, 1995.
- [Sah96] S. Sahi. Interpolation, integrality, and a generalization of Macdonald's polynomials. *Internat. Math. Res. Notices*, pages 457–471, 1996.
- [TUY89] A. Tsuchiya, K. Ueno, and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries. *Adv. Stud. Pure Math.*, 19:459–566, 1989.
- [Ver88] E. Verlinde. Fusion rules and modular transformations in 2d conformal field theory. *Nuclear Phys. B*, 300(3):360–376, 1988.
- [Wit95] E. Witten. The verlinde algebra and the cohomology of the grassmanian, “geometry, topology and physics”. *Conf. Proc. Lecture Notes Geom. Topology, IV*, pages 357–422, 1995.

